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Introduction to Geometric Invariant Theory

How to use algebraic groups to construct "good" moduli spaces.

Moduli spaces: [1]

Idea: we want to build a variety that parametrizes the objects of our problem, along with a universal family on it that gives all possible families via pullback.

Standard procedure to construct these spaces:

1. Overparametrize, to find a variety where points correspond to objects.
2. Find a group action whose orbits correspond to equivalence classes.
3. Find a suitable notion of a quotient

Examples:

1) 1-dim vector subspaces of k^n $\rightarrow$ $\mathbb{P}^n$ $\quad \mathbb{G}_m \curvearrowright \mathbb{A}^{n+1}$
2) r-dim vector subspaces of k^n $\rightarrow$ $Gr(r,n)$ $\quad GL_r(k) \curvearrowright \mathbb{A}^n$
3) degree d projective hypersurfaces of $\mathbb{P}^n$: $PGL_n(k) \curvearrowright \mathbb{P}(k[x_0, \dots, x_n]_d)$ [2, Chapter 4.2]

In each of these cases, we want to obtain a geometric quotient, i.e. a variety whose points parametrize the orbits of the action and with the quotient topology. How to do that?

Reminder - conventions:

- A variety is a reduced, separated scheme of finite type / k.
- Zariski topology
- Equivalence of categories $\{\text{fin. gen. reduced } k\text{-algebras}\} \leftrightarrow \{\text{affine varieties} / k\}$

$$A \mapsto \operatorname{Spec}(A)$$

$$\{\text{regular functions on } X\} = \mathcal{O}(X) \leftarrow X$$

Overview of algebraic groups. $/\mathbb{C}$

Def: An algebraic group G is a variety $/k$, with $\mu: G\times G \to G : (g,h)\to g\cdot h$, $\iota: G \to G : g \to g^{-1}$ morphisms of schemes, putting on its closed points a group structure.

Ex: $GL_n(k) = \{\det \neq 0\} \subset \mathbb{A}^{n^2} = \mathrm{Spec}(k[x_{ij} \mid i,j=1,..,n])$

$SL_n(k) = \{\det = 1\} \subset GL_n(k)$

$O_n(k) = \{ M \in GL_n \mid M^t\cdot M = Id_n \}$ — M is a pt $\in \mathbb{A}^{n^2}$, pol. eq'n in $\mathcal{O}(\mathbb{A}^{n^2}) = k[x_{ij} \mid i,j=1,..,n]$: $(M^tM)_{ij} = \sum_k m_{ik} m_{kj} = \delta_{ij}$

$SO_n(k) = O_n(k) \cap SL_n(k)$

$\mathbb{G}_m := k^\times \qquad \mathbb{G}_a := (k,+)$

All the group theory language comes across: subgroups, morphisms, Jordan decomposition...

Prop: An algebraic group is smooth. (generic smoothness only holds on a perfect field)

Proof: generic smoothness: $\exists U$ open of G smooth; translate it by $g.(-)$ to cover G $\square$

This is a standard argument.

Convention: we will from now on consider affine Linear algebraic groups, i.e. subgroups of some $GL_n(\mathbb{C})$.

If interested, there is a structure theorem of Chevalley saying that

Th [Chevalley]: $\forall G$ A.G, $\exists!$ maximal Linear alg. closed subgroup H: G/H is an abelian variety.

Group actions:

Def: Let G be a LAG$/\mathbb{C}$, X be a $\mathbb{C}$-variety. An action of G on X is a morphism of varieties $\cdot : G\times_{\mathbb{C}} X \to X$ that is a group action on closed points.

Ex: $\mathbb{G}_m \curvearrowright \mathbb{A}^{n+1}$: $s.(x_0,..,x_n) = (sx_0,..,sx_n)$.
$\mathbb{C}^\times$

"Are we just looking at affine varieties in $\mathbb{A}^n$? What about arbitrary ones?"

<u>Th</u>: [Equivariant embeddings]

Let G a LAG, X a affine / projective G-variety. Then there exists a G-equivariant embedding in $\mathbb{A}^N$ / $\mathbb{P}^N$,

on which G acts via a representation $G \to GL(\mathbb{A}^n)$ / $PGL(\mathbb{P}^n)$.

<u>Proof</u>: Affine case: $\mathcal{O}(X)$ is generated by $w_1, \dots, w_N$. $W := \langle w_1, \dots, w_N \rangle$ is a f.d. v.s.

Let $S(W) := \bigoplus_{i \geq 0} S^i(W)$. ① G acts linearly on $S(W)$
② The natural morphism $S(W) \twoheadrightarrow \mathcal{O}(X)$ is surjective

$\rightsquigarrow$ Gives a closed embedding $X \hookrightarrow W$ ∎

Projective case: Same thing with homogeneous generators. □

<u>Moral</u>: For "all" intents and purposes, it is enough to study matrix groups acting on vector spaces!

So, with all of this, how do we take a quotient?

<u>Def</u>: If G acts on an affine variety X, there is an induced action $G \curvearrowright \mathcal{O}(X) : g.f := f \circ g^{-1}(\cdot)$.

(Called the regular representation if X is a vector space)

If we take the ring of invariants $\mathcal{O}^G(X)$, then $\mathrm{Spec}(\mathcal{O}^G(X))$, we naturally have a surjection

$X \twoheadrightarrow \mathrm{Spec}(\mathcal{O}^G(X))$ which has topological quotient properties: (is continuous)

Th: [3, Th. 5.9] If G is a linearly reductive group acting on an affine variety X, then $\pi : X \to X/\!/G$ is surjective and gives 1-1 corr. between pts and closed orbits.

Idea of proof: orbits are all locally closed, so orbits of max dim. have smaller ones in their closure. As a result, in the closure of each maximal orbit there is a unique closed orbit.

Closure-equivalence: any two orbits $\mathcal{O}, \mathcal{O}'$ s.t. $\bar{\mathcal{O}} \cap \bar{\mathcal{O}}' \neq \emptyset$ will fail to be distinguished by G-invariant functions, and thus will map to the same point in the quotient.

<u>Th</u>: [Nagata, Mumford] Needs linear reductivity.

Problems with Spec $(\mathcal{O}^G(X))$.

Def: A LAG / $k=\bar{k}$ is reductive if its unipotent radical is trivial.
largest unipotent normal subgroup.

- Is $\mathcal{O}^G(X)$ of finite type / $\mathbb{C}$?

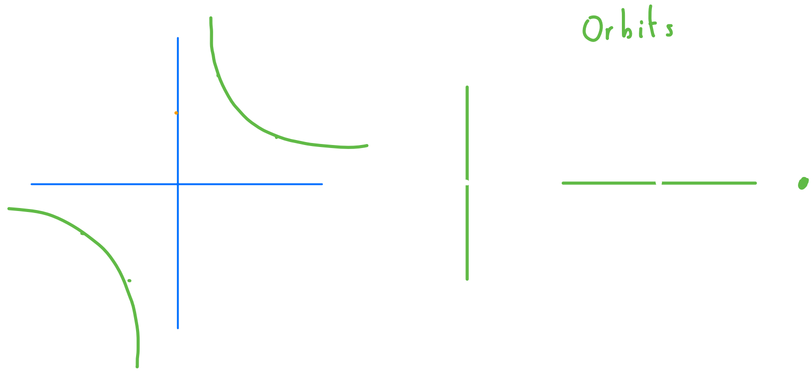
  This was Hilbert 14^{th} problem!! Answer: NO.
  But yes, if G is linearly reductive. :)

- Is the resulting variety separated?
- 'Separates closed orbits' or not?

Short version: only really separates closures of any two orbits, but inside that, nothing.

Example: $k[xy] = \mathcal{O}^G(X)$

$\mathbb{G}_m \curvearrowright \mathbb{A}^2$: $t.(x,y) \hat{=} (tx, t^{-1}y)$



| -- and . are together!

stabilizer only jumps dimension at 0. remove 0?

→ No longer separated! Need to take out these as well

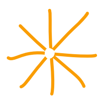
We see that we need to exclude some points to get "decent" quotients.

Stability, instability and things in between.

Def: A point in a G-variety $x \in X$ is stable if its orbit is closed and its stabilizer finite.

Then we have the following:
- X^s is open in X. of G-stable
- $X^s/\!/G$ is a geometric quotient, i.e. G-orbits in X^s $\overset{1:1}{\leftrightarrow}$ pts in $X^s/\!/G$.

Problem: $X^s \overset{?}{=} \emptyset$. In the case of $\mathbb{P}^n$, for instance, yes.

$\mathbb{G}_m \curvearrowright \mathbb{A}^{n+1}$  No points are stable!

→ We need a better theory.

The Proj quotient [3, Chapter 6]

Notation: $\mathcal{X}(G) = \mathrm{Hom}_{\underline{\mathrm{Alg}}\,\underline{\mathrm{Grp}}/k}(G, \mathbb{G}_m)$ is the group of characters of G.

$\mathcal{O}(X)^{\chi\text{-}G} = \{ f \in R \mid g.f = \chi(g).f \}$

Given $\chi \in \mathcal{X}(G)$, $G \curvearrowright X$ affine, let $\mathcal{O}(X)^{\chi\text{-}G}$ be the set of semi-invariants of weight χ.

- $\mathcal{O}(X)^{\chi\text{-}G}$ is a vector space, and $f, g \in \mathcal{O}(X)^{\chi\text{-}G} \Rightarrow f.g \in \mathcal{O}(X)^{\chi^2\text{-}G}$.

$\Rightarrow$ The ring $\bigoplus_{m \in \mathbb{Z}} \mathcal{O}(X)^{\chi^m\text{-}G}$ is graded. !

Def:
- A pt $x \in X$ is **χ-semistable** if $\exists f \in \bigoplus_m R^{\chi^n\text{-}G}$, f non constant, s.t. $f(x) \neq 0$. We write $X^{\chi\text{-}ss} \subset X$.
- $X^{\chi\text{-}ss}$ is open and G-stable. $X \setminus X^{\chi\text{-}ss}$ is the **χ-null cone**, or the **χ-unstable** pts.

need weights ≥ 0.

The **projective quotient relative to χ of X by G** is $X /\!/_\chi G := \mathrm{Proj}\left(\bigoplus_{\mathbb{Z}} R^{\chi^n\text{-}G}\right)$.

Fact: there is a morphism of varieties $X^{\chi\text{-}ss} \twoheadrightarrow X/\!/_\chi G$, which fits in the diagram

$$\begin{tikzcd}
X^{\chi\text{-}ss} \arrow[r, hook] \arrow[d] & X \arrow[d] \\
\mathrm{Proj}\left(\bigoplus_{m \in \mathbb{Z}} \mathcal{O}(X)^{\chi^m\text{-}G}\right) := X^{\chi\text{-}ss}/\!/_\chi G \arrow[r] & X/\!/G =: \mathrm{Spec}(\mathcal{O}(X)^G)
\end{tikzcd}$$

Think about $\mathbb{P}^n$:

$\mathcal{O}(\mathbb{A}^n)^{\mathbb{G}_m} = \bigoplus_{d \geq 0} k[x_0, \dots, x_n]_d$.

So $\mathrm{Proj}\left(\mathcal{O}(\mathbb{A}^{n+1})^{\mathbb{G}_m}\right) = \mathbb{P}^n$ $\square$

Remark: If $\chi = 1$ (trivial), then $\bigoplus_m \mathcal{O}(X)^{\chi^m\text{-}G} = \mathcal{O}(X)^G[u]$, so $\mathrm{Proj}(\dots) = \mathrm{Spec}(\mathcal{O}^G(X))$.

$\rightarrow$ Generalization of affine quotient!

χ - Stability

At last, we have a precise notion of stability.

A point $x \in X^{\chi\text{-}ss}$ is **χ-stable** if its orbit in $X^{\chi\text{-}ss}$ is closed (Not in X!) and $\mathrm{Stab}_G(x)$ is finite.

$\Rightarrow$ **Th:** $X^{\chi\text{-}ss}/\!/_\chi G$ is a geometric quotient.

Example: quiver moduli [4]

$Q:\ 1 \underbrace{\overset{\vdots}{\rightrightarrows}}_{m} 2$, $\underline{d} = (d_1, d_2)$ dimension vector ,

$$X = \bigoplus_{i=1}^{m} \mathrm{Hom}_k(k^{d_1}, k^{d_2}) = \bigoplus_{i=1}^{m} \mathrm{Mat}_{d_2 \times d_1}(k) . \quad \text{(a large affine space)}$$

$G = GL_{d_1}(k) \times GL_{d_2}(k)$ acts on X by base change:

$$(g_1, g_2).\alpha \coloneqq \alpha' = g_2 \alpha g_1^{-1} .$$

$$\begin{tikzcd} k^{d_1} \arrow[r, "\alpha"] \arrow[d, "g_1"'] & k^{d_2} \arrow[d, "g_2"] \\ k^{d_1} \arrow[r, "\alpha'"'] & k^{d_2} \end{tikzcd}$$

This is the simultaneous base change problem.

$\mathcal{X}(G) = \mathbb{Z}^2$, as $\mathcal{X}(GL_n) = \mathbb{Z}.\det(\cdot)$ Let us pick as stability parameter

$\theta = (d_2, -d_1)$.

What are the semi-invariants? In general, open problem (kind of).

If $\underline{d} = (1, r)$, $\theta = (r, -1)$, then the arrows $\alpha_1, \dots, \alpha_m$ are vectors in k^r.

We can visualize them in a matrix: $r\left\{ \underbrace{\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_m \end{pmatrix}}_{m} \right. =: A$, so that the action looks like

$$(g_1, g_2).A = \underset{r \times r}{\begin{pmatrix} g_2 \end{pmatrix}} \cdot \underset{r \times m}{A} \cdot g_1^{-1} \cdot \mathbb{1}_{m \times m} .$$

Assume $m > r$.

<u>Fact</u>: the semi-invariants of weight 1 are the $r \times r$ minors of A. This mean that a representation is semistable $\Leftrightarrow$ $\mathrm{rk}(A) = r$.

To visualize the quotient, we give an embedding with the generators of $\mathcal{O}(X)^{\theta \cdot G}$, the minors $\{\min_I(\cdot)\}_I$

$$\rightsquigarrow \varphi : X^{\theta\text{-ss}} /\!\!/_{\theta} G \to \mathbb{P}^{\binom{m}{r}-1} : A \mapsto [\cdots : \min_I(A) : \cdots] .$$

This is the Plücker embedding of the Grassmannian $Gr(r, m)$.

References

[1] Hoskins, Moduli spaces and GIT: old and new perspectives, arXiv: 2302.14499.

[2] Mumford, Geometric Invariant Theory.

[3] Mukai, An introduction to Invariants and Moduli.

[4] Reineke, Moduli of representations of Quivers, arXiv: 0802.2147