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Semiorthogonal decompositions for quiver moduli

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$k = \bar{k}$, char $k = 0$.

Def: A quiver is a finite oriented multigraph $Q = (Q_0, Q_1)$

Def: A representation of Q of dim. vector $d \in \mathbb{N}^{Q_0}$ is $(M_\alpha : k^{s\alpha} \to k^{t\alpha})_{\alpha \in Q_1}$

i.e. V is a point in $R(Q,d) \mathrel{\hat{=}} \bigoplus_{\alpha \in Q_1} \mathrm{Mat}_{t\alpha \times s\alpha}(k)$

$GL_d(k) \mathrel{\hat{=}} \bigtimes_{i \in Q_0} GL_{d_i}(k)$ acts on $R(Q,d)$ by base change: $g = (g_i)_{i \in Q_0}$ acts on $V = (M_\alpha)_{\alpha \in Q_1}$

$$g \cdot V \mathrel{\hat{=}} (g_{t\alpha} M_\alpha g_{s\alpha}^{-1})$$

Def: $V, W \in R(Q,d)$ are iso if they belong to the same orbit of GL_d.

$$\begin{tikzcd} \overset{d_1}{\underset{1}{\bullet}} \arrow[r, bend left, "\alpha"] \arrow[r, "\beta"] \arrow[r, bend right, "\gamma"'] & \overset{d_2}{\underset{2}{\bullet}} \end{tikzcd}$$

V is a choice of 3 $d_2 \times d_1$ matrices,

Quiver Moduli (via GIT)

The affine quotient $R(Q,d)/GL_d = \mathrm{Spec}\left(\mathcal{O}(R(Q,d))^{GL_d}\right)$.

[LB-Procesi] : $\mathcal{O}(R(Q,d))^{GL_d}$ is "generated" by cycles of Q.

$\to$ If Q is acyclic, $R/GL_d = \bullet$

<u>Def</u>: Let $\theta \in \mathbb{Z}^{Q_0}$: $\theta \cdot d = 0$. $V \in R(Q,d)$ is θ-$\left|\begin{array}{l}\text{stable}\\ \text{semistable}\end{array}\right.$ if

$\forall\, 0 \neq W \subsetneq V$, $\theta \cdot \underline{\dim}(W) \left|\begin{array}{l} < 0 \\ \leq 0 \end{array}\right.$.

[King] : "equivalent to standard GIT stability". $\ddot\smile$

$$\begin{tikzcd}
R^{\theta\text{-st}}(Q,d) \arrow[r, hook] \arrow[d] & R^{\theta\text{-sst}}(Q,d) \arrow[r, hook] \arrow[d] & R(Q,d) \arrow[d] \\
M^{\theta\text{-st}}(Q,d) \arrow[r, hook, "\text{open}"'] & \mathrm{Proj}\left(\bigoplus_{n\in\mathbb{Z}} \mathcal{O}(R(Q,d))^{n\theta}\right) \arrow[r, two heads, "\text{projective}"'] & \mathrm{Spec}\left(\mathcal{O}(R(Q,d))^{GL_d}\right).
\end{tikzcd}$$

smooth.

<u>Claim</u>: If θ-sst $\sim$ θ-st, and Q is acyclic, $M^{\theta}(Q,d)$ a smooth projective variety.

VBAC.

$C, g, r, \mathcal{L} \rightsquigarrow M_C(r, \mathcal{L})$.

- Cats of global dim 1,
- Slope stability $\rightsquigarrow$ HN filtration
- Smooth projective Fano varieties, - Pic is free Abelian
- SOD.

SOD

Given a triangulated cat $\mathcal{C}$, $\{\mathcal{C}_i\}_{i=1}^n$ of full admissible triangulated subcategories is a SOD if

- $\{\mathcal{C}_i\}$ generate $\mathcal{C}$ "classically".
- $\forall i<j, \forall k \in \mathbb{Z}$, $\mathrm{Hom}(F_j, F_i[k]) = 0$ $\forall F_i \in \mathcal{C}_i$, $F_j \in \mathcal{C}_j$.

$\mathcal{C} = \langle \mathcal{C}_1, \ldots, \mathcal{C}_n, \ldots \rangle$

For moduli of VBAC, $D^b(M_C(r, \mathcal{L}))$

$\exists$ Φ (a FMT)

$\Phi : D^b(C) \to D^b(M_C(r, \mathcal{L}))$.

(Poincaré bundle)

[Lee-Narasimhan] Φ is fully faithful : $D^b(C) \hookrightarrow D^b(M)$ (if $\gcd(r, \deg(\mathcal{L})) = 1$).

. $\mathrm{Pic}(M) = \mathbb{Z}$, + Fano $\Rightarrow$ H the ample generator of Pic is $H = -\frac{1}{e} K_M$, e is the index of M. $e=2$

If $g \geq 0$,

$$\begin{tikzcd} D^b(C) \arrow[r, hook] \arrow[d, "T"] & D^b(M) \\ D^b(C) \otimes \mathcal{O}(H) \arrow[ur] & \end{tikzcd}$$

$D^b(M) = \langle \mathcal{O}, D^b(C), \mathcal{O}(H), \mathcal{O}(H) \otimes D^b(C), \ldots \rangle$

Can we do the same for quiver moduli?

1. ~~Embedding of $D^b(C) \hookrightarrow D^b(M)$~~
2. "A good" choice of $\mathcal{O}(H)$
3. A "tool" to check SOD.

Given Q and k, kQ the path algebra of Q.

representation of Q $\Leftrightarrow$ kQ-modules.

$$D^b(Q) = \langle s_i \rangle_{i=1}^{\#Q_0}.$$

Q acyclic

If $\gcd(d)=1$, $\Pi^\theta(Q,d)$ admits a universal family.

$$\mathcal{U} = \bigoplus_{i \in Q_0} \mathcal{U}_i \qquad \mathrm{rk}(\mathcal{U}_i) = d_i .$$

$\mathcal{U}$ a left $\mathcal{O}_M$-mod structure, AND a right kQ-mod. structure.

We define

$$\Phi_{\mathcal{U}} : D^b(Q) \to D^b(\Pi^\theta(Q,d))$$
$$V \mapsto \mathcal{U} \otimes^L_{kQ} V .$$

Claim: well defined :)

[BBFR, BBFPR] $\rightsquigarrow$ under "nice" conditions, $\Phi_{\mathcal{U}}$ is fully faithful.

$$D^b(Q) \hookrightarrow D^b(\Pi^\theta(Q,d))$$
$$\langle s_i \rangle \mapsto \langle \mathcal{U}_i \rangle$$

[FRS] : $\mathrm{Pic}(\Pi^\theta(Q,d)) = \mathrm{Pic}(R(Q,d)^{\theta\text{-sst}})^{GL_d}$.

- $\mathrm{Pic} = \mathbb{Z}^{\leq \# Q_0 - 1}$
- If $\theta = \theta_{can}$, $\mathrm{Pic} = \mathbb{Z}^{Q_0 - 1}$,

$$L(\theta_{can}) \underset{\text{descends}}{\longrightarrow} \omega_\Pi^\vee$$

$$\downarrow$$

index of $\Pi^{\theta_{can}}(Q,d) = \gcd(\theta_{can}) =: e$

Let $H = \frac{-1}{e} K_\Pi$.

Q1 : Under "nice" assumptions, is

$$\langle \mathcal{O}, D^b(Q), \mathcal{O}(H), \mathcal{O}(H) \otimes D^b(Q), \dots, \mathcal{O}((e-1)H), \mathcal{O}((e-1)H) \otimes D^b(Q), \dots \rangle$$

a SOD of $D^b(\Pi^\theta(Q,d))$?

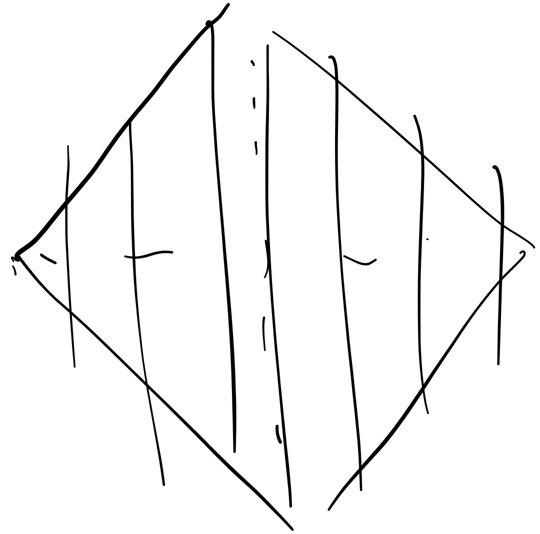
This does not always work: Hochschild homology. b_0

b_1

$\vdots$

(1) $$HH_\bullet(C) = \bigoplus_{i=1}^{r} HH_\bullet(C_i).$$

$[n]$



HH_{-1} HH_0 HH_1 ...

For VBAC, if $r=g=2$,

- $M_C(r,\mathcal{L})$ = intersection of 2 quadrics $\subset \mathbb{P}^5$
  $\rightsquigarrow \dim HH_0 = 4$ (LHS of (1))
- RHS $\geq 1+2+1+2$ ↯

For VBAC, $r=g=2$ is conjectured to be the only one not working.

$\dim M_C(r,d) = (r^2-1)(g-1)$

For quiver moduli, we know HH_0, and (Q1) gives a lower bound for RHS of (1).

$$\begin{tikzcd} \overset{1}{\bullet} \arrow[r, bend left] \arrow[r, bend right] & \overset{1}{\bullet} \end{tikzcd}$$

$\theta = (1,-1)$

$\downarrow \mathbb{P}^1$ $\rho = 3$

Q1 would give $\langle \mathcal{O}, \mathcal{U}_1, \mathcal{U}_2, \mathcal{O}(H), \mathcal{U}_1(H), \mathcal{U}_2(H),$
$\mathcal{U}_1(2H), \mathcal{U}_2(2H), \dots \rangle$

In low dim, $M_C(r,d)$ are scarce, but quiver moduli are abundant:

Lemma 3.2. *Let Q, $\mathbf{d}$ and θ be as in Assumption 2.1. If we can find a linearisation $\mathbf{a}$ for $\mathcal{U}$ such that, for all $k \geq 0$, for all $1 \leq s \leq r-1$ and for all $i, j \in Q_0$,*

$$\mathrm{H}^k(\mathrm{M}, \mathcal{U}_i^\vee \otimes \mathcal{U}_j \otimes \mathcal{O}(-sH)) = 0, \tag{24}$$

$$\mathrm{H}^k(\mathrm{M}, \mathcal{O}(-sH)) = 0, \tag{25}$$

$$\mathrm{H}^k(\mathrm{M}, \mathcal{U}_i \otimes \mathcal{O}(-sH)) = 0, \tag{26}$$

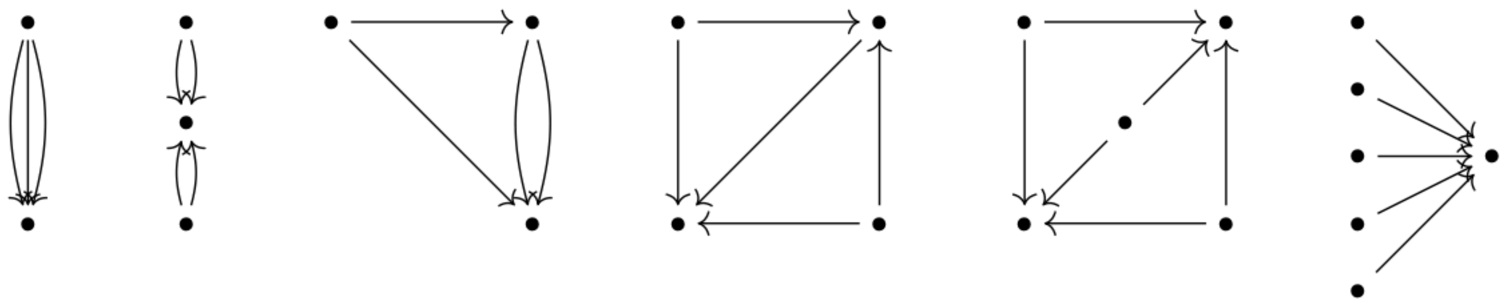
and that for all $0 \leq t \leq r-1$,

$$\mathrm{H}^k(\mathrm{M}, \mathcal{U}_i^\vee \otimes \mathcal{O}(-tH)) = 0, \tag{27}$$

then Question ~~B~~ (Q1) has a positive answer.

- Teleman quantization (Kirwan surjectivity) [HL] $\rightarrow$ tool to get $\checkmark$ $H^{\geq 1}(\cdot) = 0$
- Computations in $CH(M)$: $\rightsquigarrow$ $\chi(\cdot)$
- Serre duality.

Th [P.] (Q1) gives a full strongly exceptional collection (and SOD) for every rigid dP surface.



Q_1 $\quad$ Q_2 $\quad$ Q_3 $\quad$ Q_4 $\quad$ Q_5 $\quad$ Q_6

Table 3
Rigid del Pezzo surfaces as quiver moduli and their exceptional collections.

del Pezzo	$M^{\theta_{can}}(Q, \mathbf{d})$	$\dim HH_0(M)$	$\mathrm{rk}(\mathrm{Pic}(M))$	r	Full exc. collection.
$\mathbb{P}^2$	$M^{\theta_{can}}(Q_1, \mathbf{1})$	3	1	3	$\mathcal{O}, \mathcal{U}_1, \mathcal{U}_2$
$\mathbb{P}^1 \times \mathbb{P}^1$	$M^{\theta_{can}}(Q_2, \mathbf{1})$	4	2	2	$\mathcal{O}, \mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$
$\mathrm{Bl}_1(\mathbb{P}^2)$	$M^{\theta_{can}}(Q_3, \mathbf{1})$	4	2	1	$\mathcal{O}, \mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$
$\mathrm{Bl}_2(\mathbb{P}^2)$	$M^{\theta_{can}}(Q_4, \mathbf{1})$	5	3	1	$\mathcal{O}, \mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \mathcal{U}_4$
$\mathrm{Bl}_3(\mathbb{P}^2)$	$M^{\theta_{can}}(Q_5, \mathbf{1})$	6	4	1	$\mathcal{O}, \mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \mathcal{U}_4, \mathcal{U}_5$
$\mathrm{Bl}_4(\mathbb{P}^2)$	$M^{\theta_{can}}(Q_6, (1, \ldots, 1, 2))$	7	5	1	$\mathcal{O}, \mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \mathcal{U}_4, \mathcal{U}_5, \mathcal{U}_6$